

Diluted neural networks with adapting and correlated synapses

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We consider the dynamics of diluted neural networks with clipped and adapting synapses. Unlike previous studies, the learning rate is kept constant as the connectivity tends to infinity: the synapses evolve on a time scale intermediate between the quenched and annealing limits and all orders of synaptic correlations must be taken into account. The dynamics is solved by mean-field theory, the order parameter for synapses being a function. We describe the effects, in the double dynamics, due to synaptic correlations.

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In recent years, many models with a coupled dynamics of fast Ising spins and slow interactions have been studied to understand the simultaneous learning and retrieval in recurrent neural networks [1,2]. A major approach to this problem is replica mean-field theory with the replica number being the ratio of two temperatures characterizing the stochasticity in the spin dynamics and the interaction dynamics, respectively [3,4]. Recently, this approach was used to study coupled dynamics in the XY spin glass [5,6]; the generalization of these ideas [7] to the case of a hierarchy of subsystems with different characteristic time scales, in the Sherrington-Kirkpatrick model, interestingly leads to Parisi's solution [8]. Other approaches to coupled dynamics in neural networks are described in [9], using a discrete time master equation approach, and in [10], exploring temporal learning rules. Stochastic learning rules in diluted neural networks were considered in [11]: it was shown that in order to preserve the associative memory capability of the network, the learning rate q must be kept very small [e.g., $q = O(1/K)$, where K is the connectivity]. Moreover, in [11] the choice of a very small learning rate implied that the correlation between synaptic variables could be neglected so that the dynamics was solved by flow equations for a few macroscopic order parameters. It is the purpose of this work to reconsider coupled dynamics in diluted neural networks and keep the learning rate fixed as the connectivity K tends to ∞ . The dynamics of the network, in this limit, can be exactly solved by taking into account all the orders of correlations between synapses, the order parameter for synapses being a function on the interval $[-1,1]$. According to the argument in [11], the functioning of this model as an associative memory is questionable; we regard it as a simple model to analyze the effects due to synaptic correlations in the double dynamics.

As in [11], we consider a diluted neural network with unidirectional synapses obeying a stochastic learning mechanism [12]. The model is made of N three-state neurons $s_i = 0, \pm 1$, each connected (by binary synapses $J_{ij} = \pm 1$) to K input sites, chosen at random among the N sites. The parallel rule for updating synapses is the following: with probability q each synapse J_{ij} assumes the value $s_i s_j$ if this product is not zero; otherwise the synapse remains unchanged. A parallel stochastic dynamics with inverse temperature β is assumed for neurons, where the local field acting on neuron s_i

is given by $h_i = (\sum J_{ij} s_j)/K$, the sum being over the input neurons. The coupled dynamics consists in alternate updating of neurons and synapses. We will consider the limit $N, K \rightarrow \infty$ with $K \ll \ln N$: it is well known [13] that neurons can then be treated as independent and identically distributed stochastic variables. Moreover, we choose q constant as $K \rightarrow \infty$: q controls the ratio between the time scales over which neurons and synapses evolve and the *adiabatic* approximation is recovered by setting q to zero [14]. As a consequence, in the present case one cannot neglect the correlations among synapses.

Let us denote by s_1, s_2, \dots, s_K the input neurons and J_1, J_2, \dots, J_K the set of K input synapses for a given neuron s_0 (due to the translational symmetry the following reasoning holds for an arbitrary s_0).

We start considering the following simple situation: the synapses are independently updated by the transition matrix: $T(\mathbf{J}|\mathbf{J}') = \prod_{\alpha=1}^K \tau(J_\alpha|J'_\alpha)$, where the transition matrix for the single synapse is the following:

$$\tau = \begin{pmatrix} 1-A & B \\ A & 1-B \end{pmatrix}.$$

A good order parameter for synapses is $x = (\sum_{\alpha=1}^K J_\alpha)/K \in [-1,1]$. Indeed, denoting with $\rho_t(x)$ the probability distribution function (PDF) for x at time t , one can demonstrate (see the Appendix) that in the large- K limit the evolution of x is ruled by a deterministic Liouville operator:

$$\rho_{t+1}(x) = \int_{-1}^1 dy \delta(x - \hat{x}(y)) \rho_t(y) \quad (1)$$

with $\hat{x} = B - A + y(1 - A - B)$. The moments of ρ_t provide the synaptic correlations:

$$\langle x^p \rangle_t = \int dx x^p \rho_t(x) = \langle J^{(1)} J^{(2)} \dots J^{(p)} \rangle_t, \quad (2)$$

where the synapses $J^{(1)}, J^{(2)}, \dots, J^{(p)}$ are all different. The probability distribution, at time t , for the local field acting on neuron s_0 is

$$P_t(h) = \frac{1}{m_t} \rho_t \left(\frac{h}{m_t} \right), \quad h \in [-m_t, m_t], \quad (3)$$

where m_t is $\langle s \rangle_t$, the average magnetization of the neuronal configuration. We will denote by $Q_t = \langle s^2 \rangle_t$ the activity of neurons, satisfying $Q_t \geq m_t$ for every time t .

Let us now come back to our problem. Due to the synaptic learning rules, the values of A and B now depend on the value of s_0 . If $s_0 = 0$, then $A = B = 0$ and $\hat{x} = y$. If $s_0 = 1$, then $A = q[(Q_t - m_t)/2]$, $B = q[(Q_t + m_t)/2]$, and $\hat{x} = qm_t + y(1 - qQ_t)$. If $s_0 = -1$, then $A = q[(Q_t + m_t)/2]$, $B = q[(Q_t - m_t)/2]$, and $\hat{x} = -qm_t + y(1 - qQ_t)$. This implies that even if at time t we know x exactly (i.e., ρ_t is a δ function), at time $t+1$, x is not determined (ρ_{t+1} will generically be a convex sum of three δ 's). The full distribution ρ now plays the role of order parameter for the synaptic variables, the time evolution law being given by a mixture of three Liouville operators:

$$\begin{aligned} \rho_{t+1}(x) = & (1 - Q_t)\rho_t(x) + \frac{Q_t + m_t}{2(1 - qQ_t)} \theta \left(1 - \left| \frac{x - qm_t}{1 - qQ_t} \right| \right) \\ & \times \rho_t \left(\frac{x - qm_t}{1 - qQ_t} \right) + \frac{Q_t - m_t}{2(1 - qQ_t)} \theta \left(1 - \left| \frac{x + qm_t}{1 - qQ_t} \right| \right) \\ & \times \rho_t \left(\frac{x + qm_t}{1 - qQ_t} \right); \end{aligned} \quad (4)$$

θ is Heaviside's function.

Let us now consider the dynamics of neurons. We assume the following form for the conditional probability for neurons:

$$P(\mathbf{s}_{t+1}|h) \propto \exp \beta (h\mathbf{s}_{t+1} + a\mathbf{s}_{t+1}^2), \quad (5)$$

where \mathbf{s} , is the vector of neurons at time t , and a controls the mean activity of the network. The time evolution law for neuronal order parameters is then given by

$$\begin{aligned} m(t+1) &= \int_{-1}^1 dx \rho_t(x) \frac{2 \sinh(\beta x m_t)}{2 \cosh(\beta x m_t) + e^{-\beta a}}, \\ Q(t+1) &= \int_{-1}^1 dx \rho_t(x) \frac{2 \cosh(\beta x m_t)}{2 \cosh(\beta x m_t) + e^{-\beta a}}. \end{aligned} \quad (6)$$

These two equations, together with Eq. (4) and the initial conditions m_0 , Q_0 , and $\rho_0(x)$, solve the double dynamics for the present model.

Now we turn to analyze the flow equations. First, we consider the case of m and Q being kept constant: ρ_t tends asymptotically to the invariant distribution ρ_∞ of Eq. (4). One can easily derive a recurrence formula for the moments of the stationary distribution:

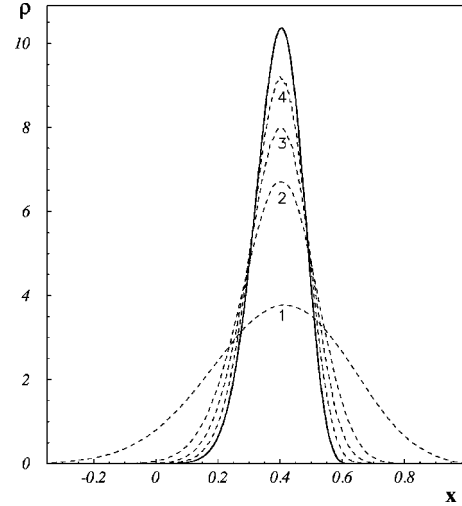


FIG. 1. The dashed lines represent the x distributions from numerical simulations for $K=20$ (1), $K=100$ (2), $K=200$ (3), $K=500$ (4), to be compared with the invariant distribution of Eq. (4), represented here by the solid line. The case $q=0.06$, $Q=0.8$, and $m=0.5$ is considered here.

$$\begin{aligned} \langle x^n \rangle_\infty = & \sum' \binom{n}{k} (1 - qQ)^{n-k} (qm)^k \langle x^{n-k} \rangle_\infty \\ & + \frac{m}{Q} \sum'' \binom{n}{k} (1 - qQ)^{n-k} (qm)^k \langle x^{n-k} \rangle_\infty, \end{aligned} \quad (7)$$

where Σ' (Σ'') is over even (odd) positive integers less than or equal to n . The invariant distribution is a δ function in the following cases. If $m=0$, then $\rho_\infty = \delta(x)$. If $m = \pm 1$, then $\rho_\infty = \delta(x-1)$, and in the adiabatic limit $q \rightarrow 0$ we have $\rho_\infty \rightarrow \delta(x - m^2/Q^2)$. In the general case the first two cumulants are given by

$$\langle x \rangle_\infty = m^2/Q^2, \quad (8)$$

which is independent of q , and

$$\langle x^2 \rangle_\infty - \langle x \rangle_\infty^2 = \frac{q}{2 - qQ} \left(\frac{m^2}{Q} - \frac{m^4}{Q^3} \right). \quad (9)$$

The last formula clearly shows how the synaptic correlations are controlled by the learning rate q . For example, in Fig. 1 the invariant distribution of Eq. (4), we numerically find, is depicted (for $q=0.06$, $Q=0.8$, and $m=0.5$). We compare it with the x distribution; over time, we find simulating a system of K synapses, evolving by the stochastic learning mechanism, where neurons s_0 and $\{s_\alpha\}$ are independently sampled with $\langle s \rangle = m$ and $\langle s^2 \rangle = Q$ at each time step. The agreement with the theoretical curve increases as K grows and it is fairly good already for $K=500$ (see Fig. 1).

The stationary regime of the coupled dynamics shows a paramagnetic phase with $m=0$ and a ferromagnetic phase with $m \neq 0$ [15]. By numerical analysis we find the transition line between the two phases in the β - a plane: in Fig. 2, our results are shown for some values of q . At fixed a , the critical temperature decreases as q is increased: the synaptic corre-

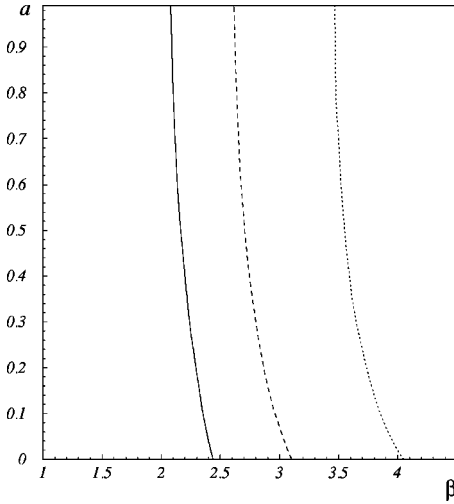


FIG. 2. In the plane β - a of parameters (see the text), the transition lines between the ferromagnetic and paramagnetic phases are depicted for $q=0$ (continuous line), $q=0.02$ (dashed line), and $q=0.05$ (dotted line).

lations seem to amplify the disordering capability of thermal noise. The two phases are separated by a first-order transition, in agreement with [3] where the para-ferro transition changes from second to first order as the influence of spins on the couplings dynamics becomes dominant.

Let us now study the role of adapting synapses in the damage-spreading phenomenon (see, e.g., [16]). For simplicity we assume two state neurons $s = \pm 1$, and we work in the disordered phase $m=0$. We assume the local fields to be

$$h_i = \frac{1}{\sqrt{K}} \sum J_{ij} s_j + B_i, \quad (10)$$

where B_i are random magnetic fields whose Gaussian distribution has variance B , and the normalization has been chosen differently from the previous case so as to have a nontrivial $K \rightarrow \infty$ limit in this case. We assume to be at zero temperature and consider two replicas of the system, subject to the same random fields and the same noise in the stochastic learning mechanism. We introduce the order parameters Δ and ϵ defined as follows: $\frac{1}{2}(1 + \Delta)$ is the probability that two corresponding synapses, in the two replicas, are equal, while $\frac{1}{2}(1 + \epsilon)$ is the probability that two corresponding neurons, in the two replicas, are equal. As in the preceding section, one easily finds that even if Δ is exactly known at a certain time, it is not determined at later times: it must be described by a probability distribution $\Gamma_r(\Delta)$, whose evolution is given by Eq. (4) with $Q=1$ and m_r replaced by ϵ_r . While keeping fixed Δ , the variables $\{J_s\}$ are equal, in the two replicas, with probability $\frac{1}{2}(1 + \Delta\epsilon)$. Therefore, the local fields in the two replicas can be written $h_1 = X + Y$ and $h_2 = X - Y$, where X and Y are random Gaussian variables with variance, respectively, $\sigma_X = (1 + \Delta\epsilon)/2 + B$, and $\sigma_Y = (1 - \Delta\epsilon)/2$. One can then easily obtain the time evolution law for ϵ :

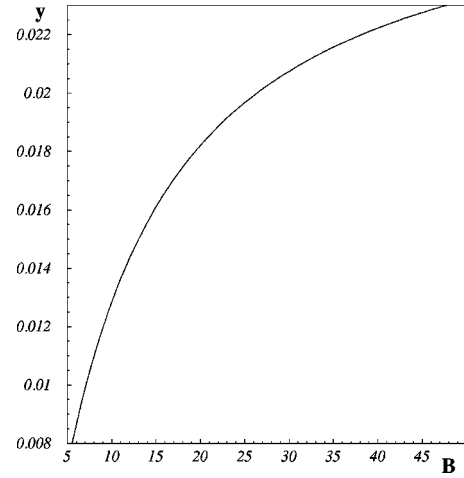


FIG. 3. Concerning the damage spreading phenomenon, $y = \partial\epsilon^*/\partial q|_{q=0}$ is depicted versus the variance of random fields, B (see the text).

$$\epsilon_{t+1} = 1 - \frac{4}{\pi} \int_{-1}^1 d\Delta \Gamma_r(\Delta) \tan^{-1} \sqrt{\frac{1 - \Delta\epsilon_t}{1 + \Delta\epsilon_t + 2B}}. \quad (11)$$

Studying damage spreading is equivalent to check the stability of the trivial fixed point $\epsilon=1$ and $\Gamma = \delta(\Delta - 1)$, corresponding to two identical replicas. We find that, for every finite B , damage spreading occurs and a nontrivial fixed point $\epsilon^* < 1$ is stable. For low values of q the stationary distribution Γ is peaked around its average ϵ^2 : approximating the \tan^{-1} by Taylor expansion at the second order around $\Delta = \epsilon^2$, the equation for the fixed point reads

$$\epsilon^* = 1 - \frac{4}{\pi} \tan^{-1} \sqrt{\frac{1 - \epsilon^{*3}}{1 + \epsilon^{*3} + 2B}} + \frac{CB\epsilon^{*2}}{\pi[(1 - \epsilon^{*3})(1 + \epsilon^{*3} + 2B)]^{3/2}}, \quad (12)$$

where $C = \langle \Delta^2 \rangle - \langle \Delta \rangle^2 = q(\epsilon^{*2} - \epsilon^{*4})/(2 - q)$ at equilibrium.

The solution ϵ^* of the equation above is the asymptotic correlation between neurons in the two replicas as a function of q . In Fig. 3, we depict $\partial\epsilon^*/\partial q|_{q=0}$ versus B . Since we find this quantity to be always positive, it follows that the synaptic correlations act against the damage-spreading phenomenon and tend to increase the correlation between the configurations of neurons in the two replicas, as one might intuitively expect.

We have described an exactly solvable model of double dynamics where synaptic correlations, arising from a stochastic learning mechanism, are important at all orders. The order parameter for synapses in the mean-field dynamical theory is a function whose evolution is given by a mixture of Liouville operators. The critical temperature for the ferromagnetic transition is found to decrease as the learning rate increases: there is a wide range of temperatures such that the system may order or not depending on the speed at which it adapts, and ordering is asymptotically achieved only if the

adaptation is sufficiently slow. We also outlined the role played by synaptic correlations in the damage-spreading phenomenon.

APPENDIX

We show the validity of Eq. (1). Using the same notation as in the text, let $P_t(\mathbf{J})$ be the PDF for synapses at time t . Then

$$P_{t+1}(\mathbf{J}) = \text{Tr}_{\mathbf{J}'} T(\mathbf{J}|\mathbf{J}') P_t(\mathbf{J}'). \quad (\text{A1})$$

It is useful to observe that, due to the symmetry of our problem, the distribution $P_t(\mathbf{J})$ will be symmetric under permutations of synapses (provided initial conditions respect the symmetry). It follows that P_t is a function of the only non-trivial invariant for permutations one can build out of K binary variables, i.e., $x = 1/K \sum_{\alpha=1}^K J_{\alpha}$.

After standard calculations [1], the probability distribution for x , $\rho_t(x)$, is found to evolve according to

$$\rho_{t+1}(x) = \int_{-1}^1 dy W_K(x,y) \rho_t(y), \quad (\text{A2})$$

where the time-independent kernel W_K is given by

$$W_K(x,y) = \frac{\text{Tr}_{\mathbf{J}} \text{Tr}_{\mathbf{J}'} \delta\left(y - \frac{1}{K} \sum J'\right) \delta\left(x - \frac{1}{K} \sum J\right) \prod_{\alpha=1}^K \tau(J_{\alpha}|J'_{\alpha})}{\text{Tr}_{\mathbf{J}'} \delta\left(y - \frac{1}{K} \sum J'\right)}. \quad (\text{A3})$$

The structure of this kernel is, in the limit $K \rightarrow \infty$,

$$W_K(x,y) = \frac{K^2}{(2\pi i)^2} \int_{-i\infty}^{i\infty} d\lambda \int_{-i\infty}^{i\infty} d\mu e^{KF(\lambda,\mu,x,y)}, \quad (\text{A4})$$

where

$$F(\lambda,\mu,x,y) = L(\lambda,\mu) - S(y) - \lambda y - \mu x, \quad (\text{A5})$$

$$S(y) = -\frac{1+y}{2} \log \frac{1+y}{2} - \frac{1-y}{2} \log \frac{1-y}{2}, \quad (\text{A6})$$

$$e^{L(\lambda,\mu)} = (1-A)e^{\lambda+\mu} + Ae^{\lambda-\mu} + Be^{-\lambda+\mu} + (1-B)e^{-\lambda-\mu}, \quad (\text{A7})$$

the time evolution for the synaptic distribution is then given by the following equation:

$$\rho_{t+1}(x) = \frac{K^2}{(2\pi i)^2} \int_{-1}^1 dy \int_{-i\infty}^{i\infty} d\lambda \int_{-i\infty}^{i\infty} d\mu e^{KF(\lambda,\mu,x,y)} \rho_t(y). \quad (\text{A8})$$

As a consequence, in the large- K limit the integral in Eq. (A8) is dominated by the physical saddle point, which means that the evolution operator W becomes, in the large- K limit, a Liouville operator, describing a deterministic evolution. The saddle point is determined by the equations $\partial F/\partial \lambda = 0$, $\partial F/\partial \mu = 0$, and $\partial F/\partial y = 0$. After a little algebra, it turns out that at the saddle point the relation $x = B - A + y(1 - A - B)$ holds. Since W_K is (by construction) normalized for every K , also the limiting kernel, as K goes to infinity, will be normalized: we can then conclude that the limiting kernel is given by $\delta(x - \hat{x}(y))$, where $\hat{x} = B - A + y(1 - A - B)$.

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